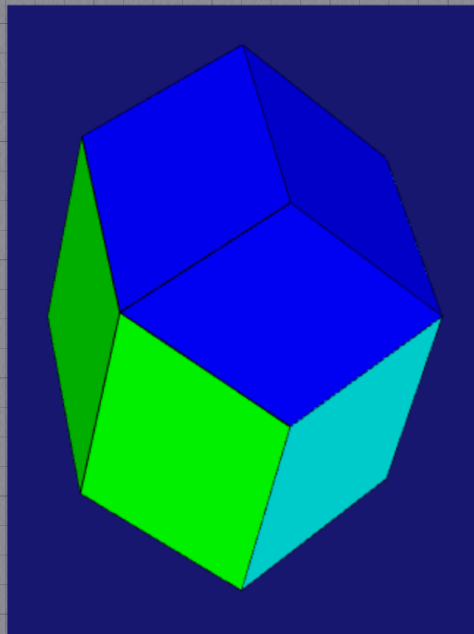


From Pentrose Tiling
to 3D Space

Acute-Angles and Obtuse Angles
Rhombic Shapes

Space Fillers



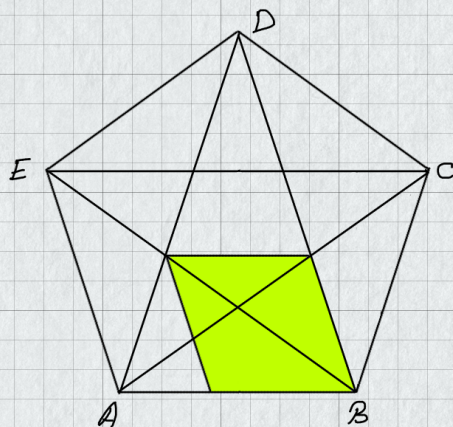
January 2022

Nick Trif

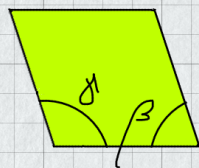
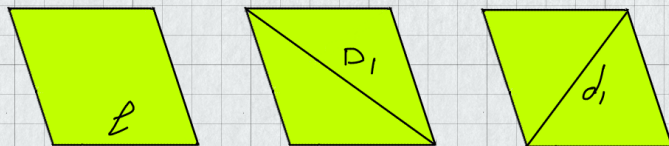
Penrose's Rhombi

In 1974 Penrose proposed these two rhombi and a matching rule as a solution to the problem of tiling the plane non-periodically. This is in fact an alternative to the pentagonal tiling published by Kepler in 1619.

Thick Rhombus

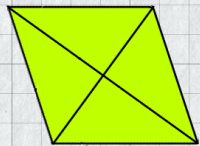


$$d_2 = \frac{1}{D_1} \quad d_1 = \frac{D_2}{\phi}$$



$$\beta = 72^\circ$$

$$\alpha = 108^\circ$$

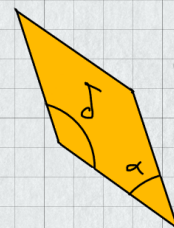
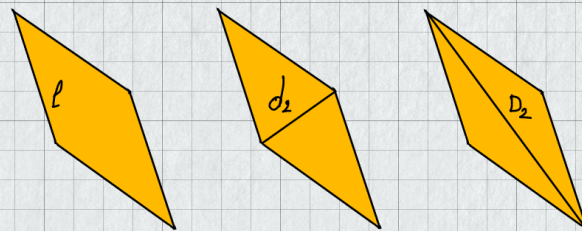
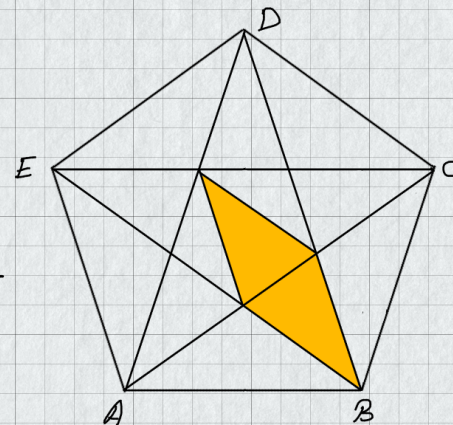


$$l_1 = 1$$

$$d_1 = \frac{\sqrt{\phi + 2}}{\phi}$$

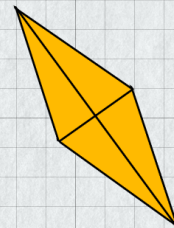
$$D_1 = \phi$$

Thin Rhombus



$$\alpha = 36^\circ$$

$$\delta = 144^\circ$$



$$l_2 = 1$$

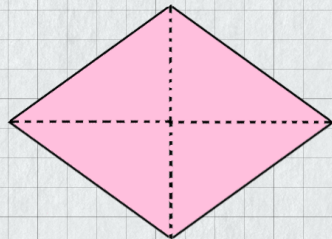
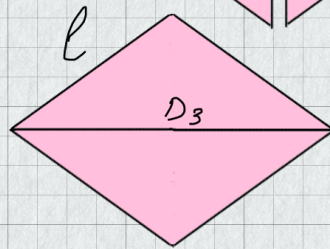
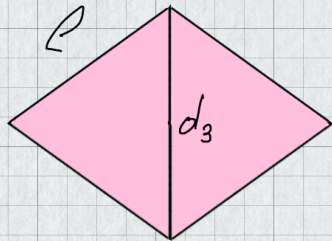
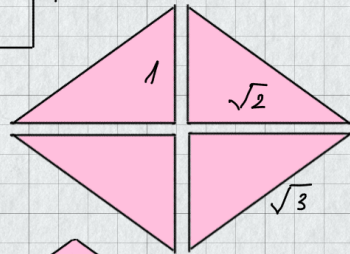
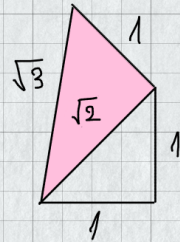
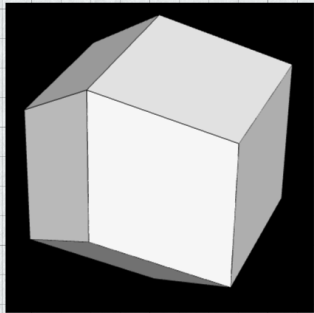
$$d_2 = \frac{1}{\phi} = \phi - 1$$

$$D_2 = \sqrt{\phi^2 + 1} = \sqrt{\phi + 2}$$

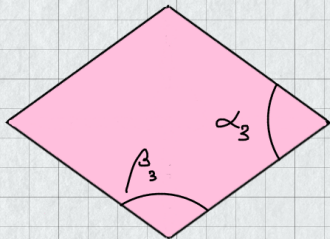
The Golden Rhombus

The Rhombic Dodecahedron Face

The Rhombic Dodecahedron was discovered by Kepler.

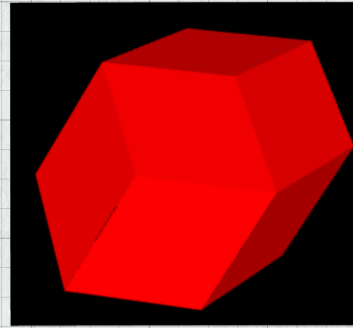


$$\begin{aligned} d_3 &= 2 \\ l &= 2\sqrt{2} \\ D_3 &= 2\sqrt{3} \end{aligned}$$



$$\begin{aligned} \alpha_3 &= 35.26^\circ \cdot 2 = 70.52^\circ \\ \beta_3 &= 109.47^\circ \end{aligned}$$

The Bilinski Dodecahedron Face

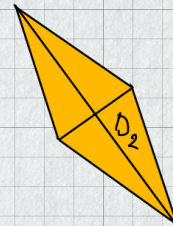
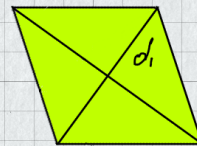


The Bilinski Dodecahedron or the Rhombic Dodecahedron of the Second Kind - rediscovered by Bilinski in 1960.

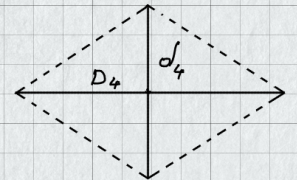
It is a space-filling polyhedron.

All 12 faces of the Bilinski Dodecahedron are golden rhombi.

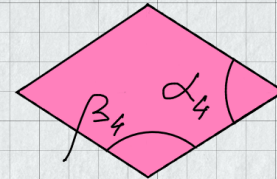
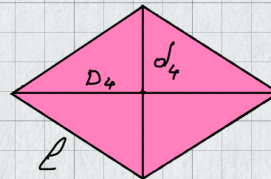
d_4	D_4
$\frac{D_4}{d_4} = \phi$	



$$\begin{aligned} d_4 &= d_1 = \frac{\sqrt{\phi+2}}{\phi} \\ D_4 &= D_2 = \sqrt{\phi+2} \end{aligned}$$



The ratio of the diagonals of the golden rhombus is the Golden Ratio ϕ .



$$\tan \frac{\alpha_4}{2} = \frac{1}{\phi}$$

$$\tan \frac{\beta_4}{2} = \phi$$

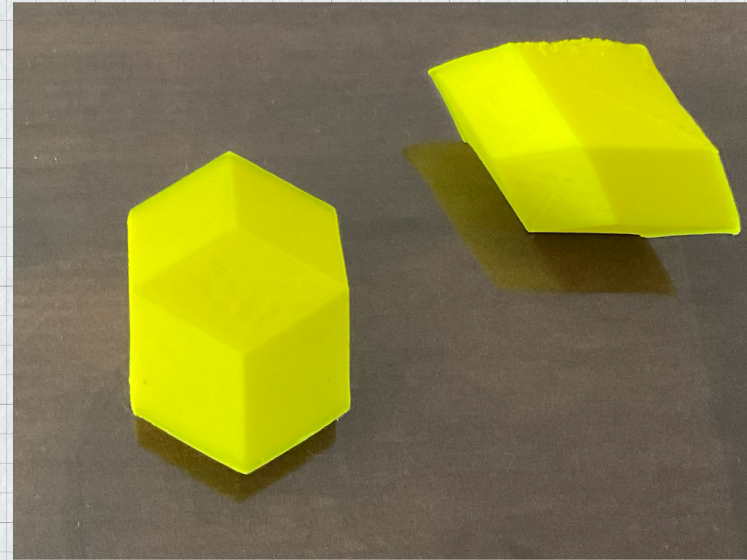
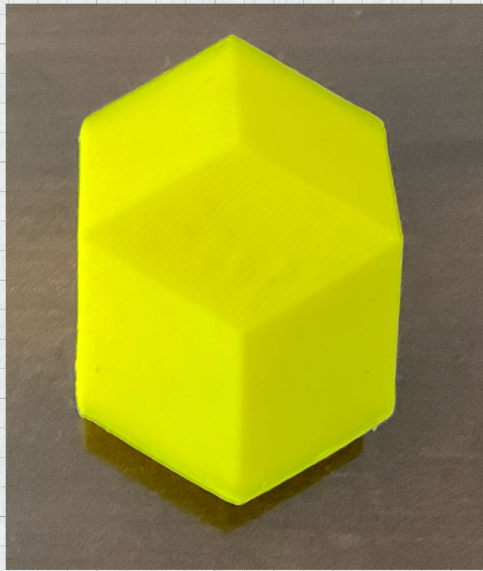
$$l^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{D_4}{2}\right)^2$$

$$\begin{aligned} \alpha_4 &= 63.43^\circ \\ \beta_4 &= 116.57^\circ \end{aligned}$$

$$l^2 = \frac{\phi+2}{4} + \frac{\phi+2}{4\phi^2} = \left(\frac{\phi+2}{2\phi}\right)^2$$

$$l_4 = \frac{\phi+2}{2\phi} = \frac{1}{2} + \frac{1}{\phi}$$

Bitinski Rhombic Dodecahedron



The Bitinski Dodecahedron or
the Rhombic Dodecahedron of the
Second Kind - rediscovered by
Bitinski in 1960.

It is a space-filling polyhedron.

All 12 faces of the Bitinski Dodecahedron are golden rhombi;

Useful Relationships Among Rhombi

Thick Rhombus

$$L_1 = 1$$

$$d_1 = \frac{\sqrt{\phi+2}}{\phi}$$

$$D_1 = \phi$$

$$d_2 = \frac{1}{D_1}$$

$$d_1 = \frac{D_2}{\phi}$$

$$d_4 = d_1$$

$$D_4 = D_2$$

$$D_4^2 = D_1^2 + 1$$

$$d_4^2 = d_2^2 + 1$$

Thin Rhombus

$$L_2 = 1$$

$$d_2 = \frac{1}{\phi} = \phi - 1$$

$$D_2 = \sqrt{\phi^2 + 1} = \sqrt{\phi + 2}$$

$$\phi - 1 = \frac{1}{\phi}$$

$$\frac{\sqrt{\phi+2}}{\phi} = \frac{\sqrt{\phi+2}}{\phi}$$

Rhombic Dodecahedron Rhombus

$$d_3 = 2$$

$$L_3 = 2\sqrt{2}$$

$$D_3 = 2\sqrt{3}$$

$$\left(\frac{d_3}{2}\right)^2 + 1 = \left(\frac{L_3}{2}\right)^2$$

$$\left(\frac{L_3}{2}\right)^2 + 1 = \left(\frac{D_3}{2}\right)^2$$

$$L_4 = \frac{\phi+2}{2\phi} = \frac{1}{2} + \frac{1}{\phi} = \phi - \frac{1}{2}$$

$$d_4 = d_1 = \frac{\sqrt{\phi+2}}{\phi}$$

$$D_4 = D_2 = \sqrt{\phi+2}$$

$$L_4 = 1.118$$

$$d_4 = 1.175$$

$$D_4 = 1.90$$

$$L_4 = 1$$

$$d_4 = 1.05$$

$$D_4 = 1.7$$

$$1^2 + 1 = (\sqrt{2})^2$$

$$(\sqrt{2})^2 + 1 = (\sqrt{3})^2$$

These are similar.

This makes the Rhombic Dodecahedron Rhombus somehow related to the Golden Rhombus.

$$\phi + 2 = \phi^2 + 1$$

$$\frac{\phi+2}{\phi^2} = (\phi-1)^2 + 1 = \phi^2 - 2\phi + 2 = 3 - \phi$$

$$\frac{\phi+2}{\phi^2} = (\phi+2)(\phi-1)^2 = (\phi+2)(\phi^2 - 2\phi + 1) = (\phi+2)(2-\phi) = 4 - \phi^2 = 3 - \phi$$

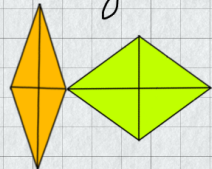
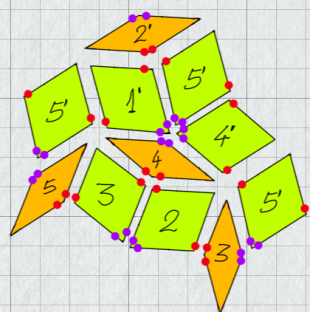
$$(\phi-1)^2 + 1 = \phi^2 - 2\phi + 2 = 3 - \phi$$

From 3D to Penrose Tiling - Case 1

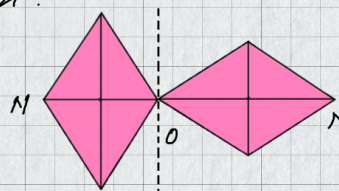
Can the Thick Rhombus and the Thin Rhombus used in the Penrose's Tiling be replaced by the projections of only one rhombus?

R.M.A. Wieringa observed that the Penrose tiling can be expressed as projections of a single rhombus from 3D space to a 2D plane.

Case 1: Consider the following region from a Penrose tiling. One can identify the following arrangement of thick and thin rhombi - relative to each other:



Let us consider next 2 golden rhombi in the following configuration:



If we rotate both rhombi, in 3D relative to the shown axis, such that the line NM is parallel with the initial plane, and

stop when the distance between the point M and the plane is 1, we get:

$$D_1^2 + 1 = D_2^2 \quad \text{on:} \quad D_4^2 = D_1^2 + 1$$

$$d_2^2 + 1 = d_1^2 \quad \text{on:} \quad d_4^2 = d_2^2 + 1$$

$$\sin \alpha = \frac{1}{D_2} = \frac{1}{\sqrt{\phi+2}}$$

$$\alpha = 31.71^\circ = \frac{\alpha_4}{2}$$

$$\beta = 58.28^\circ = \frac{\beta_4}{2}$$

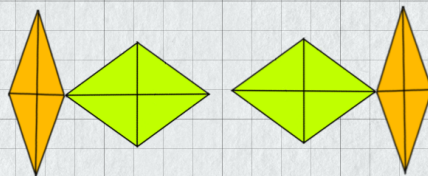
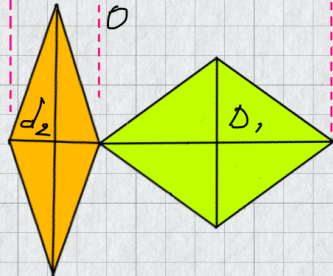
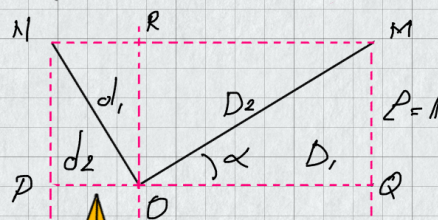
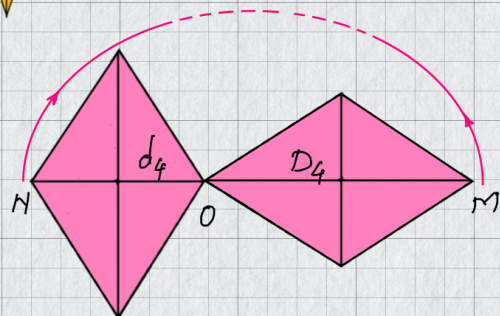
The Golden Rhombus is selected such that:

$$d_4 = d_1$$

$$D_4 = D_2$$

$$d_4 = d_1 = \frac{\sqrt{\phi+2}}{\phi}$$

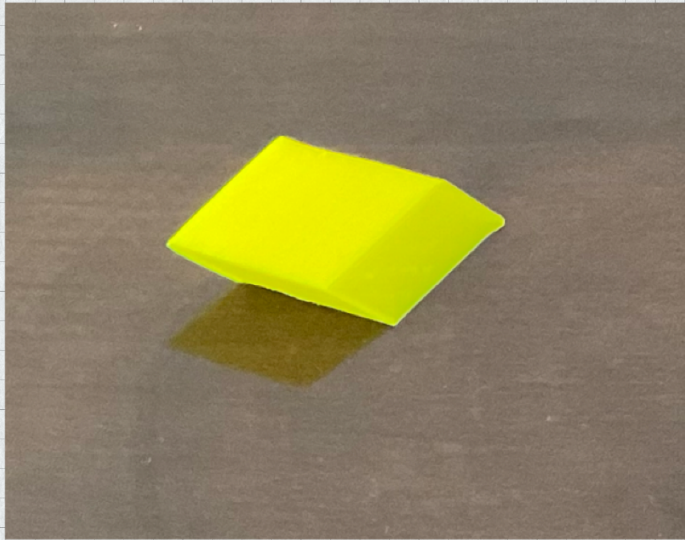
$$D_4 = D_2 = \sqrt{\phi+2}$$



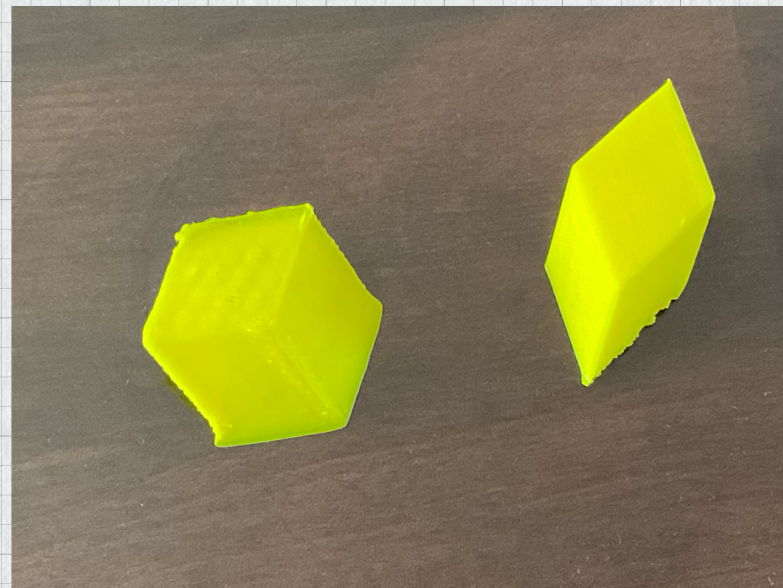
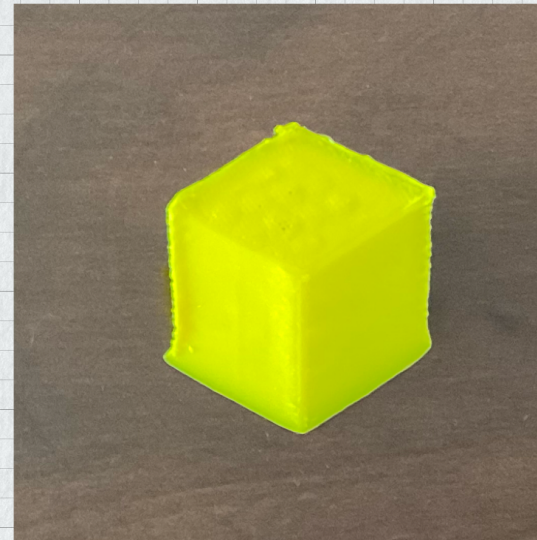
Case 1 - covers these two cases in all possible orientations.

3D Printed Models - Rhombic Shapes

Acute-Angles Rhombic Shape



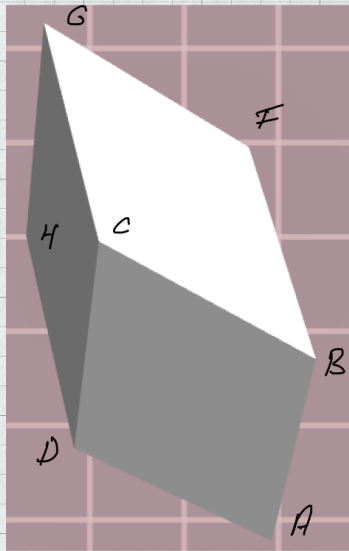
Obtuse-Angle Rhombic Shape



Accute-Angles Rhombic Shape

Two Rhombic shapes - Two modified cubes - with all faces being golden rhombi;

Accute Angles Rhombic Shape - Six golden rhombi:



$ABCD$, $ABFE$, $BCGF$, $DCGH$, $ADHE$, $EFGH$
- A and G -

- Angles BAD , BAE and EAD - acute ($\alpha_4 = 68.48^\circ$)

- Angles CBF , CGH and FGH - acute

- Points B, C, D, E, F and H - two obtuse angles ($\beta_4 = 114.57^\circ$) and one acute angle:

B: ABC and ABF - obtuse; CBF - acute;

C: BCG and DCG - obtuse; BCD - acute;

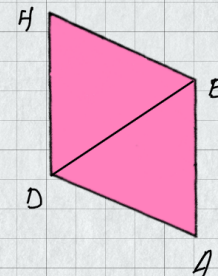
D: ADC and ADH - obtuse; CDH - acute;

...

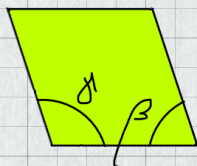
Consider a stack of Accute-Angles rhombic shapes, such that:

$AEDH$ sits on top of $BFEC$ The base of this stack is the golden rhombus $AEDH$:

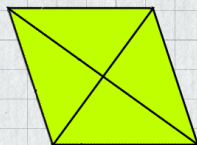
Rotating the stack around the short diagonal DE by an angle $\alpha = \frac{\alpha_4}{2} = 34.24^\circ$ will produce a projection of the golden rhombus into a Pentrose thick rhombus:



DE - short diagonal;
 $DE = d_1$



$\beta = 72^\circ$
 $\alpha = 108^\circ$



$$L_1 = 1$$

$$d_1 = \frac{\sqrt{\phi + 2}}{\phi}$$

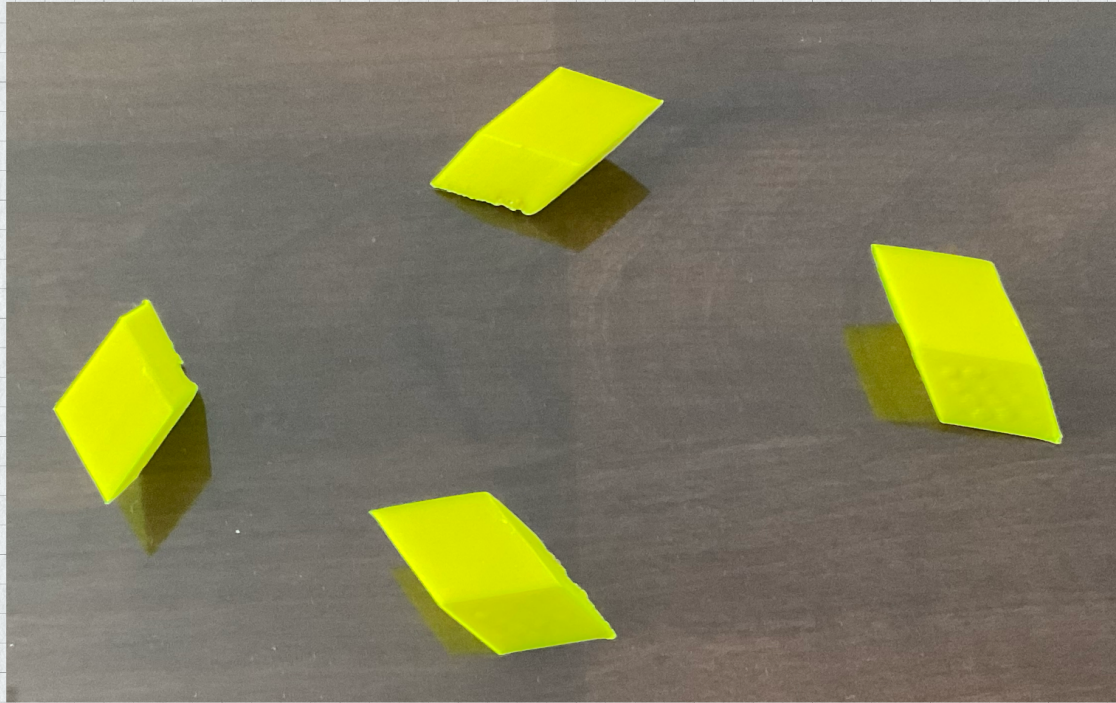
$$D_1 = \phi$$

The rotation can be done clockwise or counter clockwise.

The initial z-coordinates of A, D, E and H are zero.

The final z-coordinates of D and E are zero, and $\pm \frac{1}{2}$ of A and H.

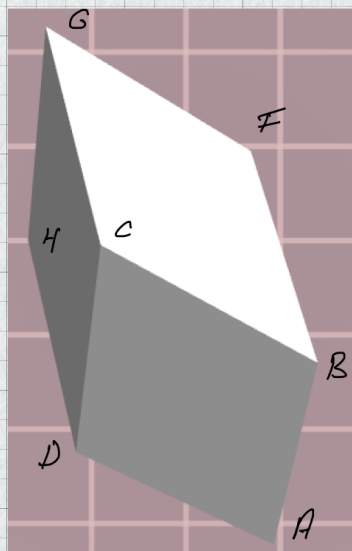
Acute-Angles Rhombic Shapes



The shape is a "modified" cube.
All 6 faces are golden rhombi.
Three faces meet in each vertex.

2 vertices formed by 3 acute-angles meeting together; $\alpha = 63.43^\circ$
4 vertices formed by 2 obtuse and one acute angles meeting; $\beta = 116.57^\circ$
Diagonal for the faces: d_1 and D_1 .

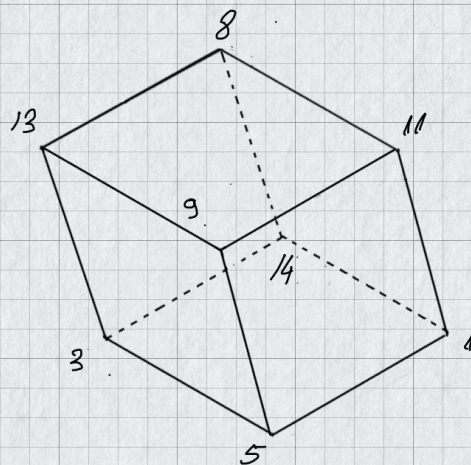
The Acute-Angles Rhombic Shape Attributes



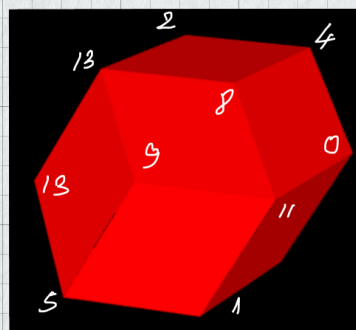
Vertices: x, y, z , coordinates:

		x	y	z
0		0	1	1
1		0	1	-1
2		0	-1	1
3		0	-1	-1
4		0	0	ϕ^2
5		0	0	$-\phi^2$
6		ϕ	0	ϕ
7		ϕ	0	$-\phi$
8		$-\phi$	0	ϕ
9		$-\phi$	0	$-\phi$
10		ϕ	1	0
11		ϕ	-1	0
12		$-\phi$	1	0
13		$-\phi$	-1	0
14		0	0	$\phi-1$

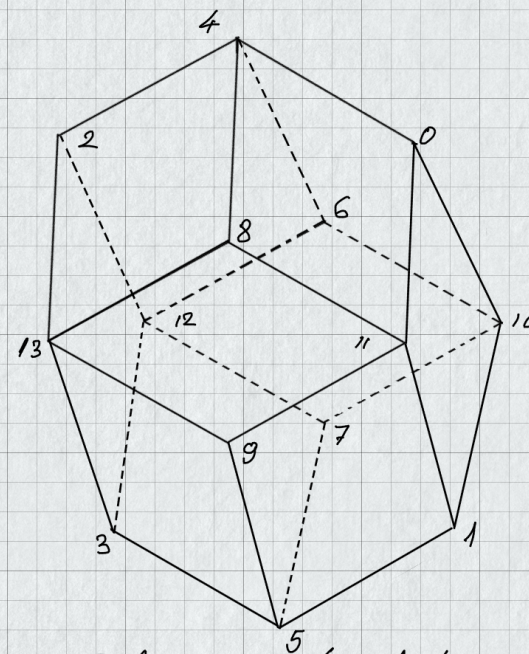
The connection between the vertices of the Acute-Angle Rhombic Shape and the vertices of the Bilinski Rhombic Dodecahedron:



Acute-Angles
Rhombic Shape.



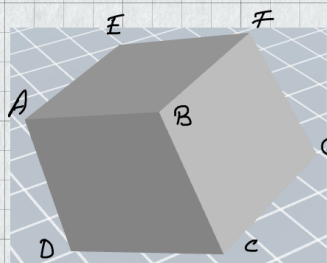
Bilinski
Dodecahedron



Bilinski Dodecahedron

Obtuse - Angles Rhombic Shape

Six golden rhombi can also be joined together to form a second kind of a modified cube. The six golden rhombi are:



$ABCD, ABFE, BCGF, CDHG, ADHE, EFGH$

- B and H : ABC, CBF, ABF - obtuse angles ($\beta_1 = 116.57^\circ$);

DHE, EHG, GHD - obtuse angles;

- A, C, D, E, F, G - two acute angles and one obtuse angle:

A: BAD and BAE - acute; EAD - obtuse;

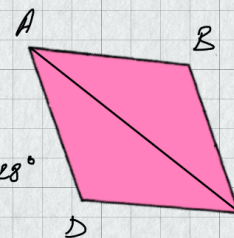
C: BCG and BCD - acute; DCG - obtuse;

D: ADH and CDH - acute; ADC - obtuse;

Again, we can consider a stack of Obtuse - Angles Rhombic Shapes such that:

$ABCD$ sits on top of $EFGH$.

The base of the stack is a golden rhombus $ABCD$:

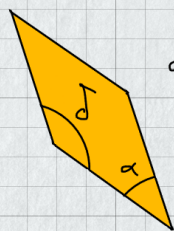


AC - Long diagonal

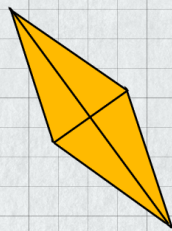
$AC = D_4$

Rotating the stack around the long diagonal AC by an angle $\beta = \frac{\beta_1}{2} = 58.28^\circ$ will produce a projection of the golden rhombus into a pentagon

thin rhombus. The rotation can be done clockwise or counterclockwise.



$\alpha = 36^\circ$
 $\delta = 144^\circ$



$D_2 = 1$

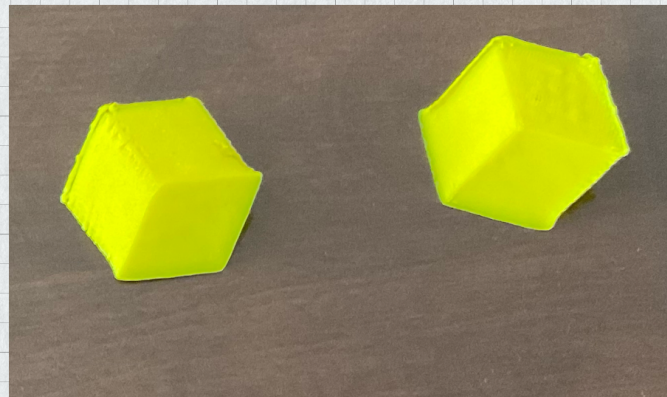
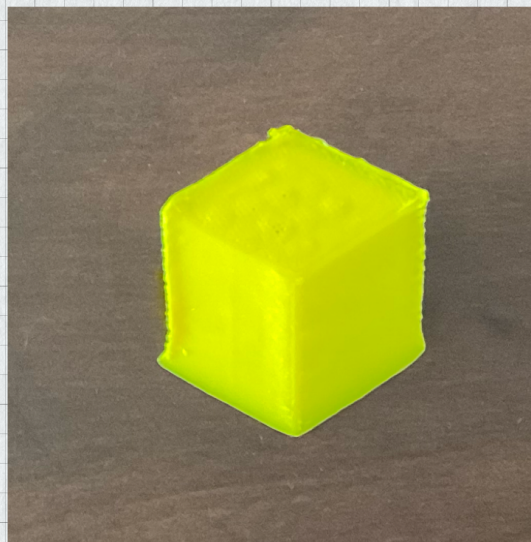
$d_2 = \frac{1}{\phi} = \phi - 1$

$D_2 = \sqrt{\phi^2 + 1} = \sqrt{\phi + 2}$

The initial z -coordinates of A and C are zero. After the rotation they will remain zero.

The initial coordinates of B and D are also zero. After the rotation they become $\pm \frac{1}{2}$ - depending on the rotation. z -coordinates are relative to the diagonal AC .

Obtuse-Angles Rhombic Shape



The shape is a "modified" cube.

All 6 faces are golden rhombi.

Three faces meet in each vertex.

2 vertices formed by 3 obtuse-angles meeting together;

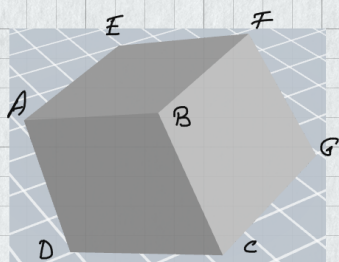
$$\beta = 116.57^\circ$$

4 vertices formed by 2 acute and one obtuse angles meeting;

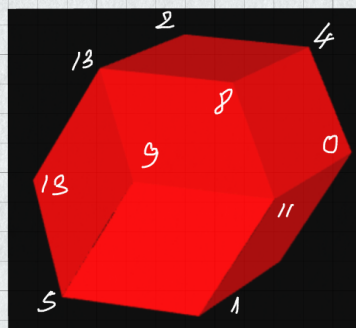
$$\alpha = 63.43^\circ$$

Diagonal for the faces: d_1 and D_1 .

The Obtuse-Angles Rhombic Shape



Obtuse-Angles Rhombic Shape

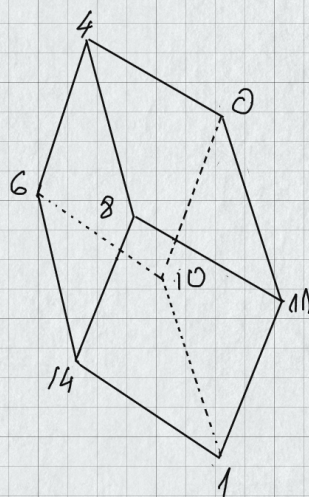


Bilinski Dodecahedron

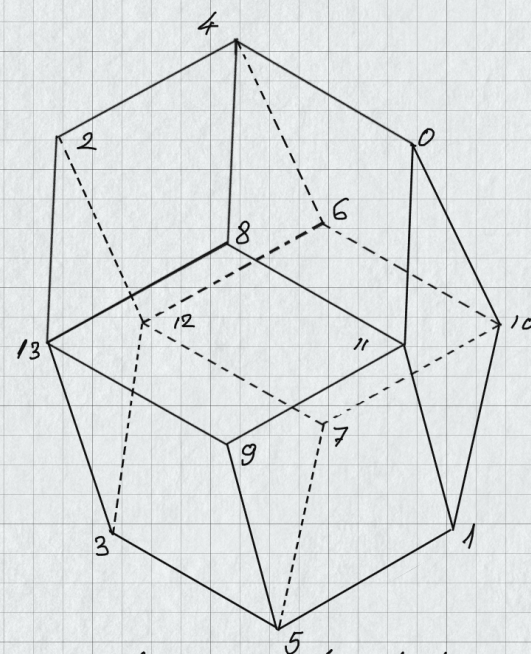
Vertices: x, y, z , coordinates:

	x	y	z
0	0	1	1
1	0	1	-1
2	0	-1	1
3	0	-1	-1
4	0	0	ϕ^2
5	0	0	$-\phi^2$
6	ϕ	0	ϕ
7	ϕ	0	$-\phi$
8	$-\phi$	0	ϕ
9	$-\phi$	0	$-\phi$
10	ϕ	1	0
11	ϕ	-1	0
12	$-\phi$	1	0
13	$-\phi$	-1	0
14	0	0	$\phi-1$

The connection between the vertices of the Obtuse-Angle Rhombic Shape and the vertices of the Bilinski Rhombic Dodecahedron:



Obtuse-Angles Rhombic Shape



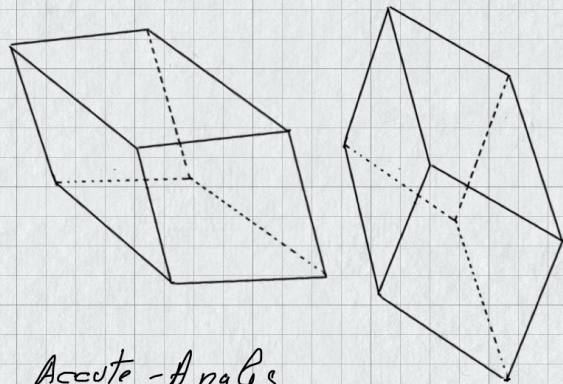
Bilinski Dodecahedron

3D Projections of Acute-Angles Rhombic Shapes

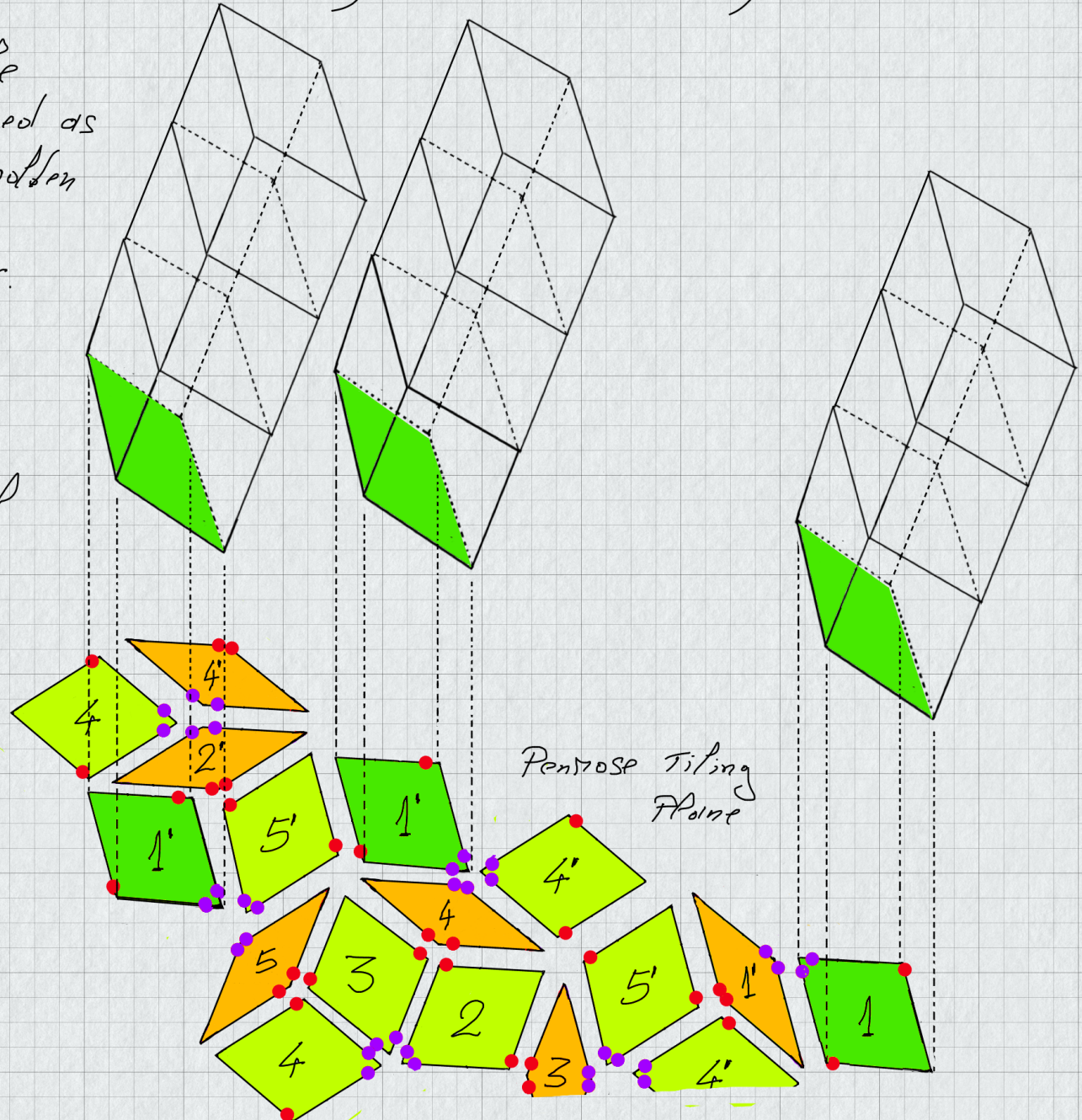
Every thick rhombi from the Penrose Tiling can be imagined as being the projection of the golden rhombus of a stack of Acute-Angles Rhombic Shapes.

Three stacks and their projections are represented here.

All the stacks are parallel with each other - they do not intersect.



Acute-Angles
Rhombic Shapes

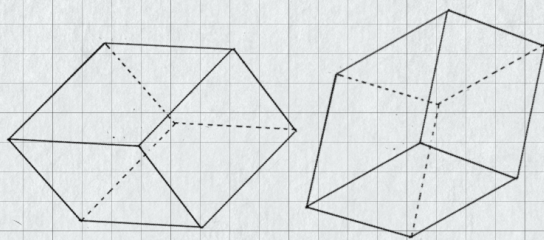


3D Projections of Obtuse-Angles Rhombic Shapes

Every thin rhombi from the Penrose Tiling can be imagined as being the projection of the golden rhombus of a stack of Obtuse-Angles Rhombic Shapes.

Two stacks and their corresponding projections are represented here

All the stacks are parallel with each other - they do not intersect.

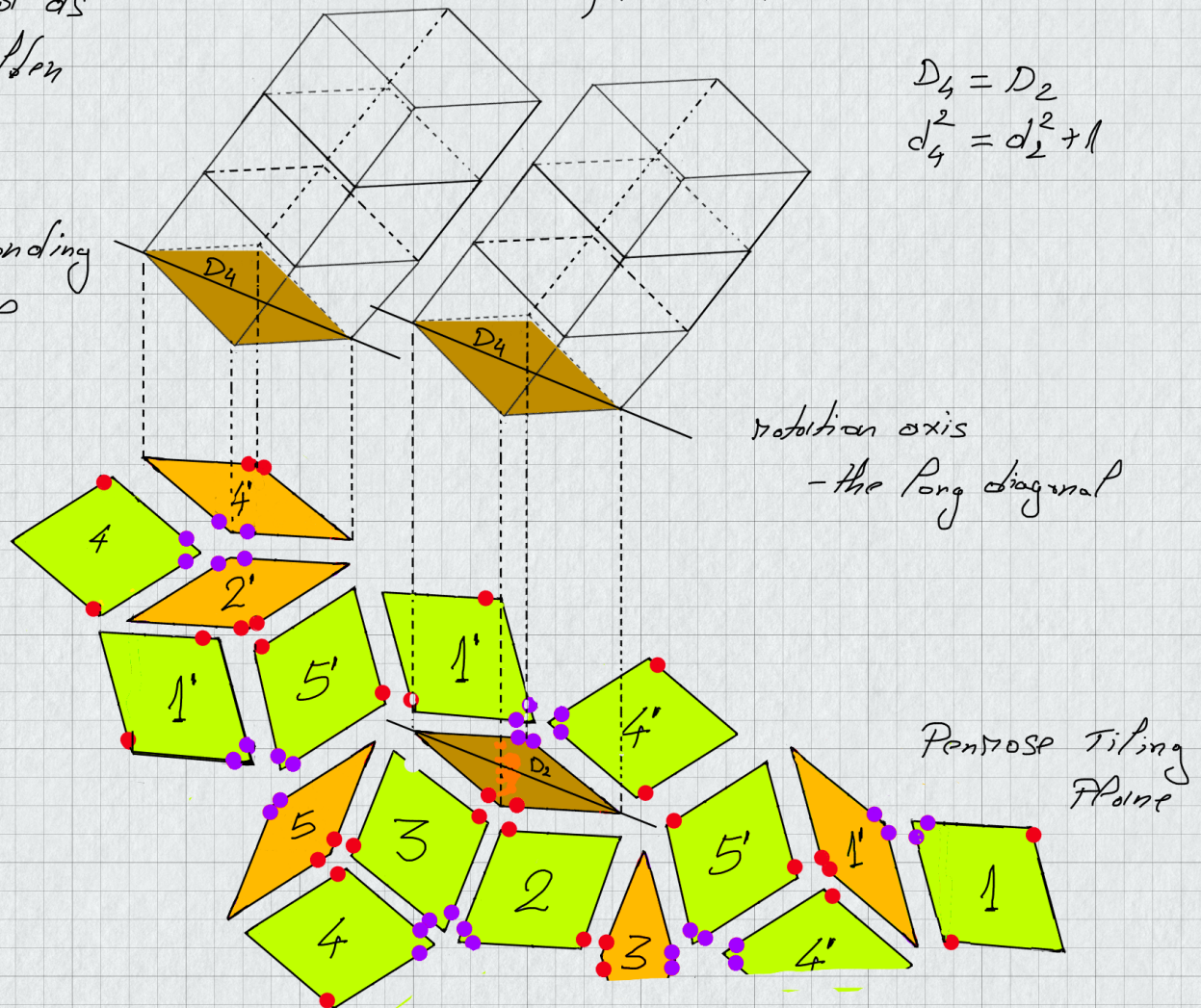


Obtuse-Angles
Rhombic Shapes

The length of the long diagonal is preserved.

$$D_4 = D_2$$

$$d_4^2 = d_2^2 + 1$$



3D Projections of Rhombic Shapes

Thick rhombi are projections of Acute-Angles Rhombic shapes rotated along the short diagonals with an angle $\alpha = \frac{\alpha_4}{2} = 31.71^\circ$

Thin rhombi are projections of Obtuse-Angles Rhombic shapes rotated along the long diagonals with an angle $\beta = \frac{\beta_4}{2} = 58.28^\circ$

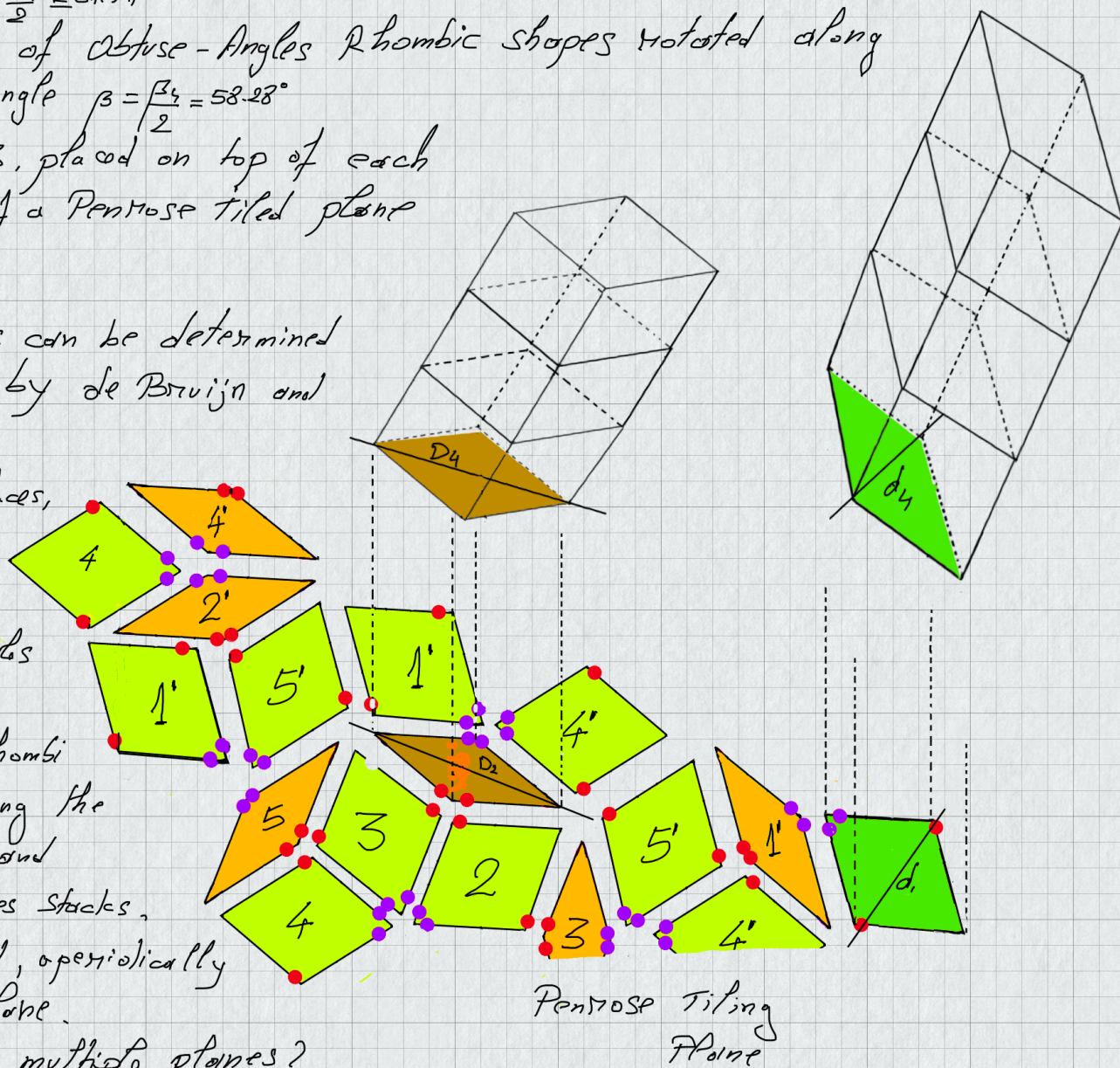
Stacks of rhombic shapes, placed on top of each thick and thin rhombus of a Penrose tiled plane fill up the 3D Space.

The heights of the stacks can be determined by the method proposed by de Bruijn and R.M.A. Wieringa.

Wieringa roof is a 3D surfaces, made up from golden rhombi, place on top of a Penrose tiled plane, at specific angles and heights.

By considering the golden rhombi of the Wieringa roof as being the bases of the Acute-Angles and Obtuse-Angles Rhombic Shapes stacks, the 3D space, can be filled, aperiodically along the considered plane.

Can this be expanded to multiple planes?



R. M. A. Wieringel Roof - The Heights of the Rhombic Shapes Stacks

MATHEMATICS

Proceedings A 84 (1), March 20, 1981

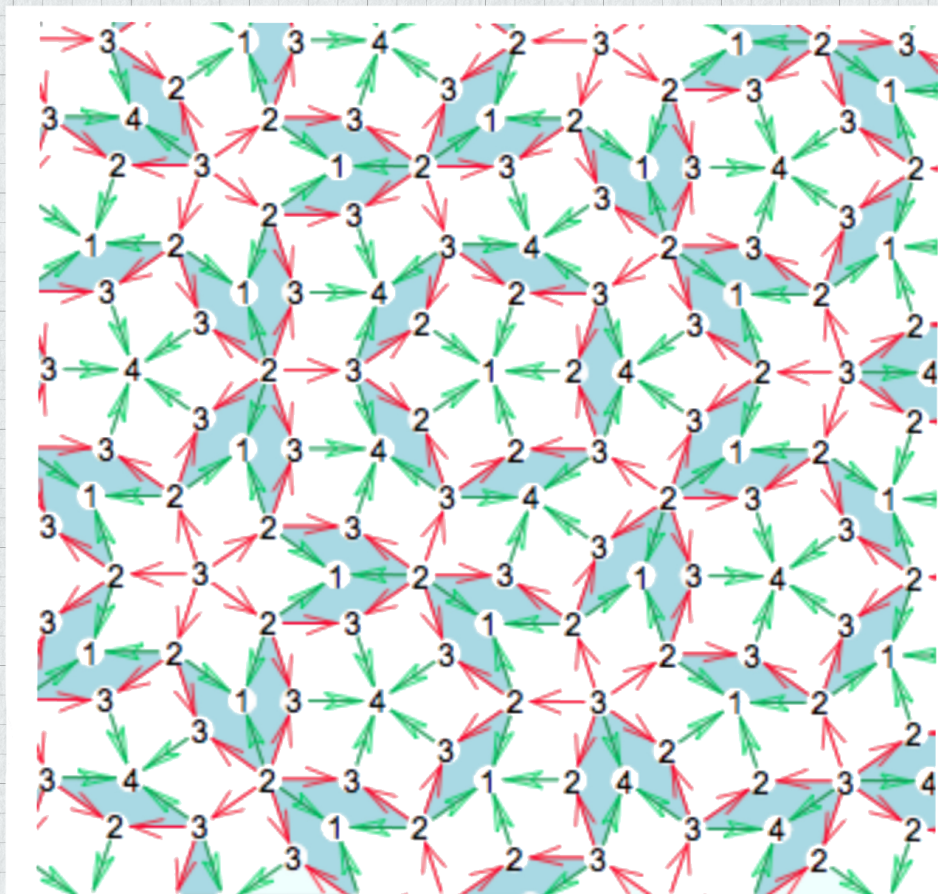
Algebraic theory of Penrose's non-periodic tilings of the plane. II

by N.G. de Bruijn

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Dedicated to G. Pólya

Communicated at the meeting of October 26, 1980



Hence, a thick rhombus either has index values 1 and 3 at the 72° angles and 2 at the 108° angles, or 2 and 4 at the 72° angles and 3 at the 108° angles. A thin rhombus either has index values 1 and 3 at the 144° angles and 2 at the 36° , or 2 and 4 at the 144° and 3 at the 36° . Edges connecting a point of index 3 to a point of index 2 are colored red, edges connecting a 1 to a 2 or a 3 to a 4 are green. Green arrows point from 2 to 1 or from 3 to 4. Orienting red edges is similar but a bit more involved.

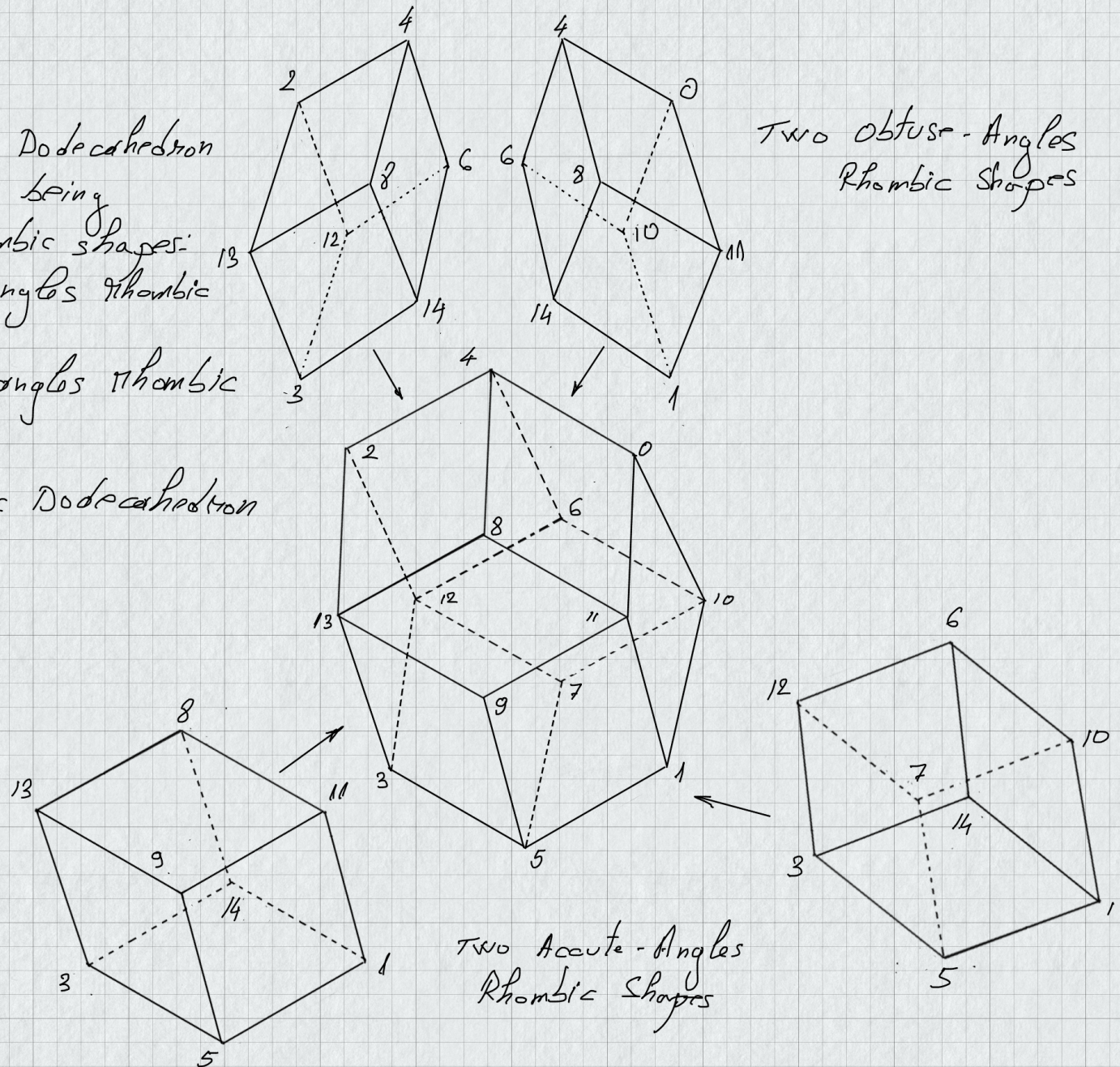
From Andrej Treibergs presentation: Penrose Tiling

Biłinski Dodecahedron - From Four Components

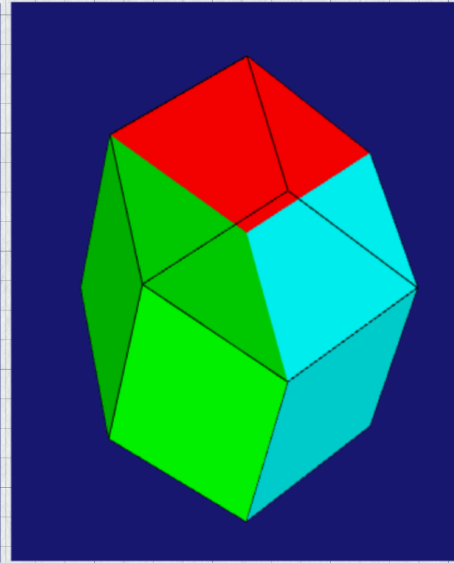
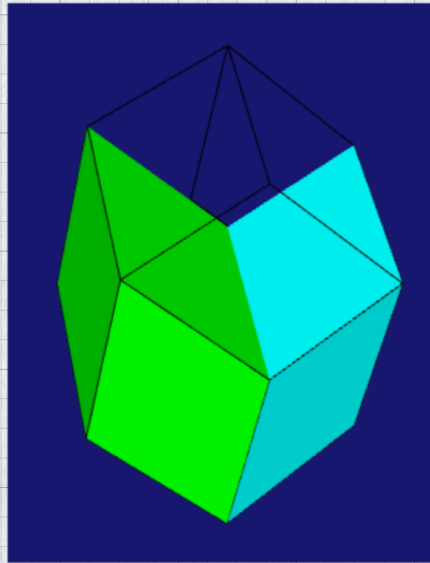
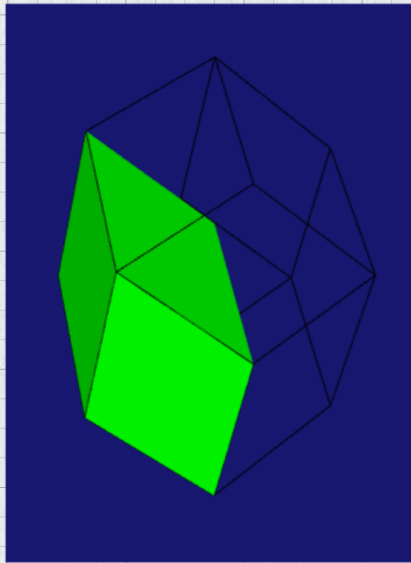
The Biłinski Rhombic Dodecahedron can be imagined as being composed from 4 rhombic shapes:

- two acute-angles rhombic shapes; and
- two obtuse-angles rhombic shapes.

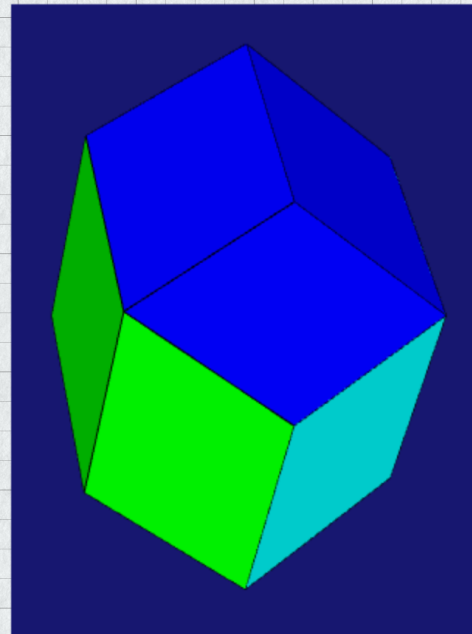
The Biłinski Rhombic Dodecahedron is a space-filler.



Bilinski Dodecahedron - Components

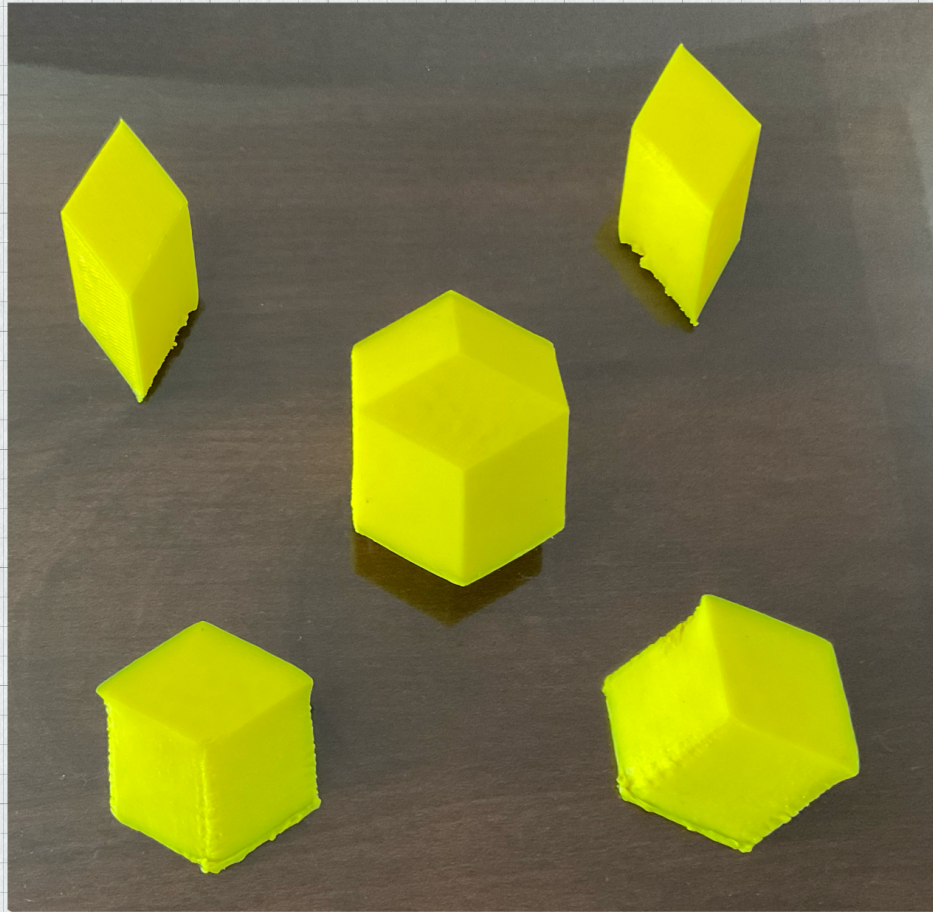


A 3D Viewer showing the four components of the Bilinski Rhombic Dodecahedron is available on our site: photonicsstructures.co



Two acute-angles rhombic shapes + two obtuse-angles rhombic shapes can form a Bilinski Rhombic Dodecahedron.

The Four Components of the Bilinski Dodecahedron

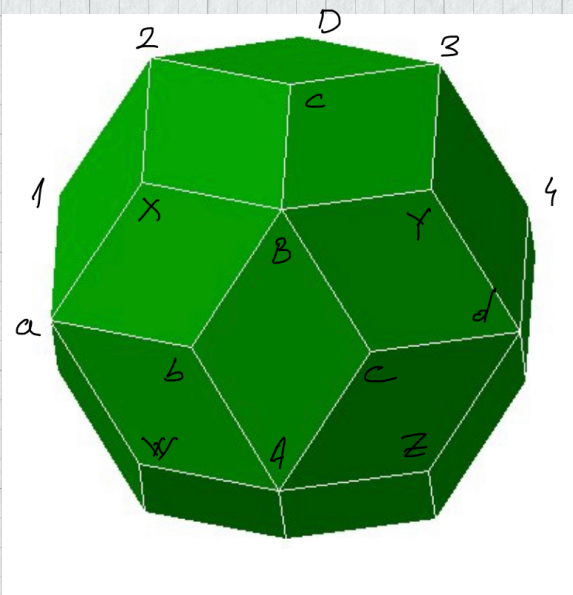


Two Acute-Angles
Rhombic Shapes

Two Obtuse-Angles
Rhombic Shapes

3D Printed Models

Rhombic Triacontahedron



30 Golden Rhombi

32 Vertices

62 Edges

3 mutually orthogonal octagons:

1, 2, 3, 4, ... golden octagon 1 - along the x-axis;
 A, B, C, D, ... golden octagon 2 - along the y-axis;
 a, b, c, d, ... golden octagon 3 - along the z-axis;

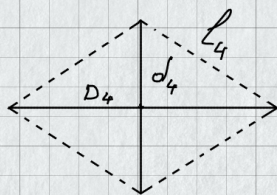
These three mutually orthogonal octagons determine 24 vertices.

The remaining 8 vertices form a cube:

XYZW... - cube

Let us consider the following golden rhombus:

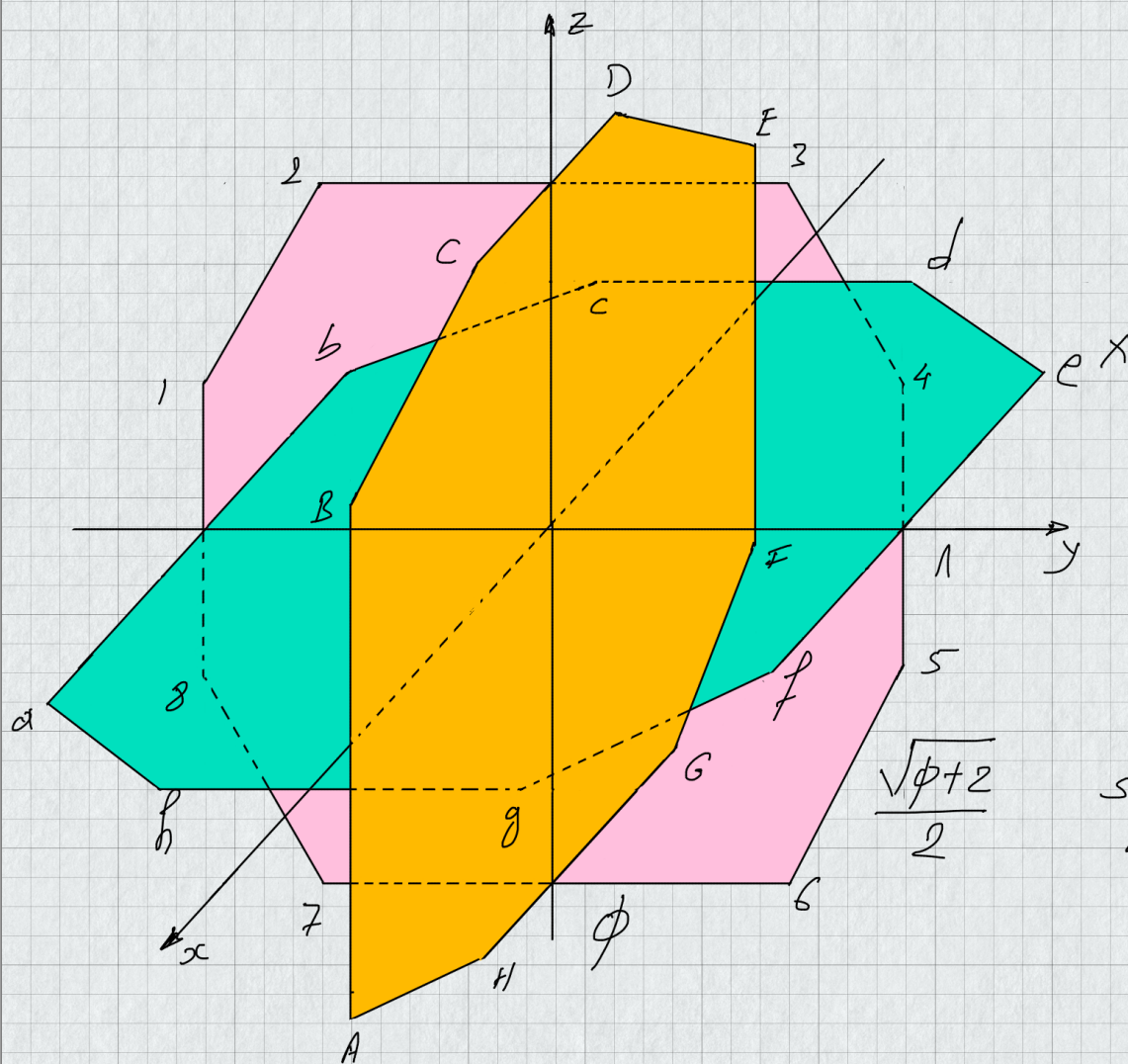
$$L_4 = \frac{\sqrt{2+\phi}}{2} ; d_4 = 1 ; D_4 = \phi$$



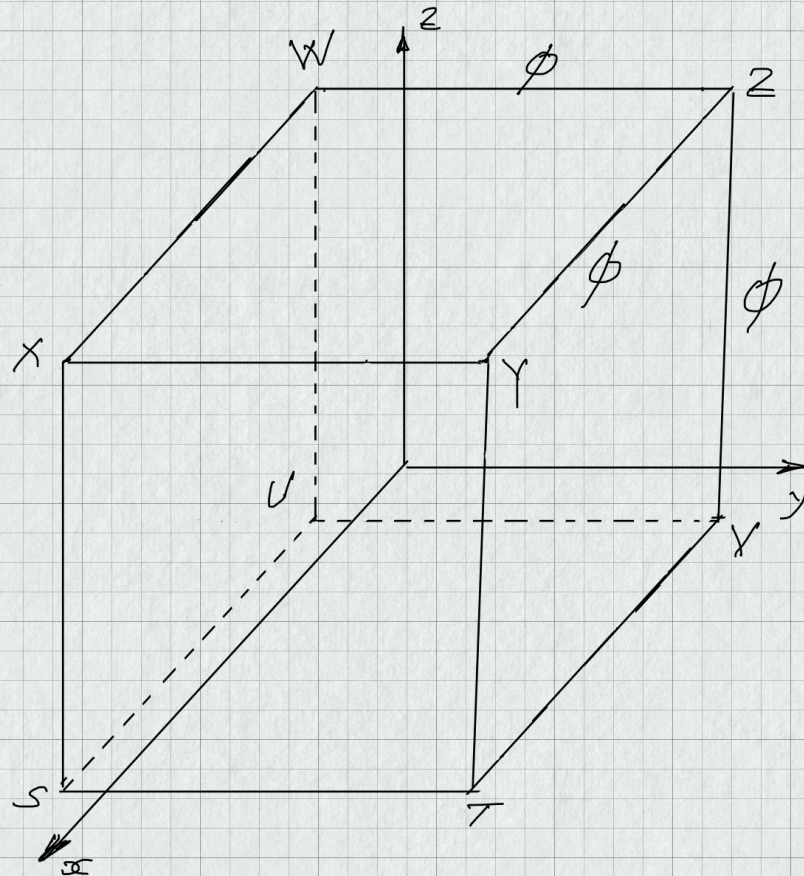
The Rhombic Triacontahedron can be imagined as a structure in the CPS Space.

Using Fibonacci, a sequence of objects can be constructed. The ideal Rhombic Triacontahedron is the result of this sequence.

The Vertices of the Rhombic Triacuboctahedron



Three mutually orthogonal
Octagons - 24 Vertices



Cut - 8 vertices

The Golden Octagon

Under a close examination one can spot 3 mutually orthogonal octagons defining 24 vertexes of the Rhombic Triacontahedron.

Each octagon can be obtained from a square using the following two conditions:

$$\begin{cases} x^2 + y^2 = \frac{2+\phi}{4} \\ 2x + \phi = 2y + 1 \quad 2y - 2x = \phi - 1 \end{cases}$$

$$y - x = \frac{1}{2\phi} \quad y = x + \frac{1}{2\phi}$$

$$x^2 + \left(x + \frac{1}{2\phi}\right)^2 = \frac{2+\phi}{4}$$

$$x^2 + x^2 + \frac{1}{\phi}x + \frac{1}{4\phi^2} = \frac{2+\phi}{4}$$

$$8\phi^2 x^2 + 4\phi x + 1 - 2\phi^2 - \phi^3 = 0$$

$$8\phi^2 x^2 + 4\phi x - 2(2\phi - 1) = 0$$

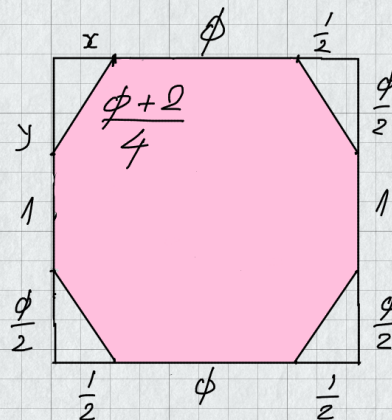
$$4\phi^2 x^2 + 2\phi x - (2\phi + 1) = 0$$

$$\Delta = 4\phi^2 + 16\phi^2(2\phi + 1) = 4\phi^2(8\phi + 5)$$

$$x = \frac{-2\phi + 2\phi\sqrt{8\phi+5}}{8\phi^2} = \frac{\sqrt{8\phi+5}}{4\phi} - \frac{1}{4\phi}$$

$$y = x + \frac{1}{2\phi} = \frac{\sqrt{8\phi+5}}{4\phi} + \frac{1}{4\phi}$$

Verification: $x^2 + y^2 = \frac{1}{4} + \frac{\phi^2}{4} = \frac{\phi^2 + 1}{4} = \frac{\phi + 2}{4} \quad \rho = \frac{\sqrt{2+\phi}}{4}$



$$x = \frac{1}{2}, \quad y = \frac{\phi}{2}$$

$$\phi^2 = \phi + 2 \cdot \frac{1}{2} \quad \phi^2 = 1 + 2 \cdot \frac{\phi}{2}$$

$$1 - 2\phi^2 - \phi^3 = 1 - 2(\phi+1) - \phi(\phi+1) = 1 - 2\phi - 2 - \phi^2 - \phi = -2(2\phi+1)$$

The golden octagon:

$$\frac{F_n}{F_{n-1}} + 1 = \frac{F_n + F_{n-1}}{F_{n-1}} = \frac{F_{n+1}}{F_{n-1}}$$

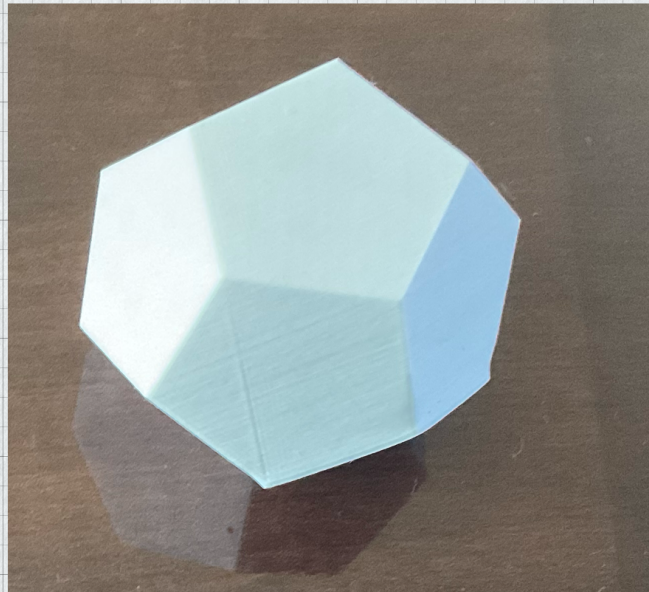
$$1, 1, 2, 3, 5, 8, 13, 21, 34$$

$$1, \frac{8}{5}, \frac{13}{5}, \frac{21}{8}, \frac{34}{13}, \dots \phi^2$$

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \phi^2$$

Planes in Space - Pentagonal Lattice

Dihedral angle $= \beta_4 = 116.57^\circ$
 δ_4



Dihedral angle of the
 zcoshedron is 138.13°
 138.13°

i_4

$$\frac{i_4 - d_4}{2} = 10.81$$

